which completes the proof.

Editor's comment. Digby Smith remembered that the following problem proposed by Jack Garfunkel and George Tsintsifas appeared in the August–September 1982 issue (Vol. 8, no. 7, p. 210) of ***Crux*** and a solution given by Vedula N. Murty appeared in the November 1983 issue (Vol. 9, no. 9, p. 282):

$$\frac{4}{9}\sum \sin B \sin C \leq \prod \cos\frac{B-C}{2} \leq \frac{2}{3}\sum \cos A.$$

Smith gave a proof by first showing that $2L \leq \sum \sin B \sin C$, which together with the above inequality yields the result.

4004. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = 2$. Prove that

$$\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)} \geq 1.$$

We received 16 correct submissions. We present 3 solutions.

Solution 1, by Arkady Alt.

Since by Cauchy's Inequality

$$\sum_{cyc} \frac{x^5}{yz\left(x^2+y^2\right)} = \sum_{cyc} \frac{x^6}{xyz\left(x^2+y^2\right)} \geq \frac{\left(x^3+y^3+z^3\right)^2}{\sum_{cyc} xyz\left(x^2+y^2\right)},$$

it suffices to prove the inequality

$$\frac{\left(x^3+y^3+z^3\right)^2}{\sum_{cyc} xyz\left(x^2+y^2\right)} \geq 1.$$

We have the following equivalences:

$$\frac{\left(x^3+y^3+z^3\right)^2}{\sum_{cyc} xyz\left(x^2+y^2\right)} \geq 1 \iff \left(x^3+y^3+z^3\right)^2 \geq 2xyz\left(x^2+y^2+z^2\right)$$
$$\iff \left(x^3+y^3+z^3\right)^2 \geq xyz\left(x+y+z\right)\left(x^2+y^2+z^2\right),$$

where the latter inequality holds because by AM-GM Inequality

$$x^3+y^3+z^3 \geq 3xyz$$

and by Chebyshev's Inequality

$$x^3+y^3+z^3 \geq \frac{(x+y+z)\left(x^2+y^2+z^2\right)}{3}.$$

Solution 2, by Michel Bataille.

Let $a = \frac{x}{2}$, $b = \frac{y}{2}$ and $c = \frac{z}{2}$. With these notations, we are required to prove

$$\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2} \geq \frac{abc}{2} \tag{1}$$

under the conditions $a, b, c > 0$ and $a + b + c = 1$.

The Cauchy-Schwarz inequality gives

$$\left(\frac{a^6}{a^2+b^2} + \frac{b^6}{b^2+c^2} + \frac{c^6}{c^2+a^2}\right)((a^2+b^2)+(b^2+c^2)+(c^2+a^2)) \geq (a^3+b^3+c^3)^2.$$

Hence, (1) will follow if we prove

$$\frac{(a^3+b^3+c^3)^2}{a^2+b^2+c^2} \geq abc.$$

Since $abc \leq \frac{a^3+b^3+c^3}{3}$, it is sufficient to show that

$$3(a^3+b^3+c^3) \geq a^2+b^2+c^2.$$

Now, the latter follows from

$$\begin{aligned}
3(a^3+b^3+c^3) &\geq 2(a^3+b^3+c^3)+3abc \\
&= a^3+b^3+c^3+(a^3+b^3+c^3+3abc) \\
&\geq a^3+b^3+c^3+ab^2+a^2b+bc^2+b^2c+ca^2+c^2a \quad \text{(Schur's ineq.)} \\
&= (a^2+b^2+c^2)(a+b+c) = a^2+b^2+c^2 \quad \text{(since } a+b+c=1\text{)}
\end{aligned}$$

so we are done.

Solution 3, by Oliver Geupel.

By hypothesis $x + y + z = 2$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left(\frac{x^5}{yz(x^2+y^2)} + \frac{y^5}{zx(y^2+z^2)} + \frac{z^5}{xy(z^2+x^2)}\right)(x+y+z)xyz(x^2+y^2+z^2) \\
&= \left(\sum_{\text{cyc}} \frac{x^5}{yz(x^2+y^2)}\right)\left(\sum_{\text{cyc}} xyz(x^2+y^2)\right) \geq (x^3+y^3+z^3)^2.
\end{aligned}$$

By the power mean inequality, it holds

$$\left(\frac{x^3+y^3+z^3}{3}\right)^{1/3} \geq \left(\frac{x^2+y^2+z^2}{3}\right)^{1/2} \geq \frac{x+y+z}{3} \geq (xyz)^{1/3}.$$

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Putting together we obtain

$$\begin{aligned}&\frac{x^5}{yz(x^2+y^2)}+\frac{y^5}{zx(y^2+z^2)}+\frac{z^5}{xy(z^2+x^2)}\\&\qquad\geq\frac{(x^3+y^3+z^3)^2}{(x+y+z)xyz(x^2+y^2+z^2)}\\&\qquad=\frac{(x^3+y^3+z^3)^{1/3}}{x+y+z}\cdot\frac{x^3+y^3+z^3}{xyz}\cdot\frac{(x^3+y^3+z^3)^{2/3}}{x^2+y^2+z^2}\\&\qquad\geq 3^{-2/3}\cdot 3\cdot 3^{-1/3}=1.\end{aligned}$$

Hence the result. By the equality condition of the power mean inequality, the equality holds if and only if $x=y=z=2/3$.

4005. *Proposed by Michel Bataille.*

Let a,b,c be the sides of a triangle with area F. Suppose that some positive real numbers x,y,z satisfy the equations

$$x+y+z=4 \quad \text{and}$$
$$2xb^2c^2+2yc^2a^2+2za^2b^2-\left(\frac{4-yz}{x}\,a^4+\frac{4-zx}{y}\,b^4+\frac{4-xy}{z}\,c^4\right)=16F^2.$$

Show that the triangle is acute and find x,y,z.

We present the proposer's solution — no others were submitted.

The second equation gives

$$\begin{aligned}&(xyz)(16F^2)\\&=xyz(2xb^2c^2+2yc^2a^2+2za^2b^2)-yz(4-yz)a^4-zx(4-zx)b^4-xy(4-xy)c^4\\&=(a^2yz+b^2zx+c^2xy)^2-(4a^4yz+4b^4zx+4c^4xy)\\&=\left(\frac{x}{2}(b^2z+c^2y)+\frac{y}{2}(c^2x+a^2z)+\frac{z}{2}(b^2x+a^2y)\right)^2-(4a^4yz+4b^4zx+4c^4xy).\end{aligned}$$

Since $t\mapsto t^2$ is a convex function and $x+y+z=4$, Jensen's inequality yields

$$\begin{aligned}&\frac{x}{4}(b^2z+c^2y)^2+\frac{y}{4}(c^2x+a^2z)^2+\frac{z}{4}(b^2x+a^2y)^2\\&\qquad\geq\left(\frac{x}{4}(b^2z+c^2y)+\frac{y}{4}(c^2x+a^2z)+\frac{z}{4}(b^2x+a^2y)\right)^2\qquad(1)\end{aligned}$$

and it follows that

$$\begin{aligned}&(xyz)(16F^2)\\&\leq x(b^2z+c^2y)^2+y(c^2x+a^2z)^2+z(b^2x+a^2y)^2-(4a^4yz+4b^4zx+4c^4xy)\\&=a^4yz(y+z-4)+b^4zx(z+x-4)+c^4xy(x+y-4)+xyz(2b^2c^2+2c^2a^2+2a^2b^2)\\&=xyz(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\\&=(xyz)(16F^2).\end{aligned}$$